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PARAMETRIC ANALYSIS OF TWO-PHASE FLOW INSTABILITY IN A CHANNEL WITH INLET AND OUTLET HYDRAULIC RESISTANCES

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The mechanism of low-frequency self-oscillating instability of a one-dimensional two-phase flow in a channel with inlet and outlet hydraulic resistances is considered. The mechanism is based on the sensitivity of the inlet flow rate of the liquid to the pressure variation inside the channel and the sensitivity of the pressure to the variation of the outlet gas flow rate (with a constant mass rate of the liquid–gas phase transition per unit volume). A spectral analysis of the stability of the steady solution of the boundary-value problem for a hyperbolic-type nonlinear system of equations is performed within the framework of a two-velocity model of a gas–liquid flow. Parametric boundaries of the region of instability are obtained. The existence of self-oscillations in this range of parameters is supported by a numerical solution of the unsteady boundary-value problem.

In various fields of engineering, use is made of devices whose principle of operation consists in that a liquid continuously flows to the inlet and complete liquid-to-gas conversion occurs inside the device. The gas is withdrawn from it via hydraulic resistance, which maintains the necessary pressure level inside the device or accelerates the gas flow to create a jet thrust. This can be illustrated by liquid jet engines and catalytic gas generators of aerospace systems and by tubular vapor generators for boilers and chemical reactors.

The instability of the stationary mode of operation in some ranges of the parameters, which causes periodic variations of the pressure, gas flow rate at the outlet, liquid flow rate at the inlet, and dimensions of the two-phase zone is an interesting phenomenon characteristic of the operation of such devices, but this phenomenon is extremely undesirable in practice.

Although the technical devices (tubes, reactors with a granular catalyst layer, and combustion chambers) and the physicochemical processes (evaporation, combustion, and catalytic decomposition of liquids) occurring in them are different, it is assumed that common mechanisms of instability exist for a certain class of observed oscillations. One can study the onset of such oscillations within the framework of a rather general mathematical model.

In the present paper, we consider a mechanism of instability that is based on (1) strong sensitivity of the inlet liquid flow rate to the pressure variation inside the device, (2) strong sensitivity of the pressure to the variation of the outlet gas flow rate, and (3) a constant mass rate of the liquid–gas phase transition per unit volume. The period of oscillations is much longer than the time of propagation of acoustic waves, i.e., pressure oscillations occur almost in the same phase throughout the volume, thus playing the role of negative feedback for an oscillatory system. It is important here that the variation in the outlet gas flow rate delays in time relative to the variation in the liquid flow rate at the inlet of the device.

A similar mechanism of oscillations is considered in the theory of the instability of liquid-propellant rocket engines for low-frequency oscillations [1]. Results of a parametric analysis of the instability were obtained using a mathematical model based on the hypothesis of a stepwise burning curve. According to this hypothesis, for all particles in the chamber there is a delay time during which the liquid remains unchanged.

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and after this period ends, the liquid instantaneously becomes a gas. An advantage of this model is the fact that mathematical analysis is simple and physical understanding of the instability mechanism is straightforward. However, this idealization is too strong for studying the operation of concrete devices and it is necessary to introduce characteristics of a two-phase flow.

In the literature on the dynamics of vapor–liquid flows in heated channels [2], the manifestation of a similar instability mechanism is called density-wave oscillations, which are usually analyzed within the framework of a one-dimensional model of a two-phase flow, with hydraulic resistance concentrated at the channel's inlet and outlet. However, spectral stability analysis is performed under the assumption that the gas-phase density is constant, i.e., the density of the two-phase mixture is determined only by the volumetric gas fraction. Such an approximate method of instability analysis leads to larger errors the larger the volumetric gas fraction in the flow, and it is not applicable at all if there is a purely gas-phase flow segment in the channel, which exerts a marked effect on the pressure dynamics. The latter case is characteristic of reactors for catalytic gas generation.

Stegasov et al. [3] considered the problem of the onset of this type of instability within the framework of a one-dimensional model of a two-phase flow with allowance for the compressibility of the gas phase and the presence of a gas-phase flow segment. The results obtained were, however, limited to the case of a pseudohomogeneous model of a two-phase flow (equality of the liquid and gas velocities).

In the present paper, we analyze a two-velocity model of a gas-liquid flow in which the liquid velocity changes simultaneously throughout the channel, depending on the inlet velocity. Here the liquid can be regarded as a solid jet with a transverse cross section decreasing along the length owing to the phase transition. This idealization allows one to perform an exact spectral analysis of the stability of the steady solution of the boundary-value problem for a hyperbolic-type nonlinear system of equations within the framework of the two-velocity model of a gas-liquid flow.

Formulation of the Problem. The mathematical model of a one-dimensional gas-liquid flow in a channel includes the equations ($0 < z < L$ and $t > 0$)

$$\frac{\partial \rho}{\partial t} + \frac{\partial G}{\partial z} = 0; \tag{1}$$

$$\frac{\partial}{\partial t}(\rho_{\text{liquid}}\beta) + \frac{\partial}{\partial z}(\rho_{\text{liquid}}\beta u_{\text{liquid}}) = -W \quad \text{for} \quad \beta > 0; \tag{2}$$

$$\frac{\partial u_{\text{liquid}}}{\partial z} = 0; \tag{3}$$

$$\frac{\partial P}{\partial z} = 0; \tag{4}$$

$$\frac{P}{\rho_{\text{gas}}} = C = \text{const}; \tag{5}$$

$$\rho = \beta\rho_{\text{liquid}} + (1-\beta)\rho_{\text{gas}}, \qquad \rho_{\text{liquid}} = \text{const}; \tag{6}$$

$$G = \beta\rho_{\text{liquid}}u_{\text{liquid}} + (1-\beta)\rho_{\text{gas}}u_{\text{gas}} \tag{7}$$

and the boundary conditions

$$z = 0: \qquad \beta = 1; \tag{8}$$

$$z = 0: \qquad G = \Phi(P); \tag{9}$$

$$z = L: \qquad G = \Psi(P). \tag{10}$$

Here L is the length of the channel (in meters); z and t are the coordinates in length (in meters) and time (in seconds), ρ, ρ_{liquid}, and ρ_{gas} are the densities of the two-phase mixture, liquid, and gas, respectively (in kg/m^3), G is the mass flow rate of the two-phase mixture per unit cross-sectional area of the channel (in kg/m$^2\cdot$sec), β is the volume fraction of the liquid in the mixture, u_{liquid} and u_{gas} are the liquid and gas velocities, respectively (in m/sec), W is the mass rate of the liquid–gas phase transition per unit volume of the channel (in $\text{kg/sec}\cdot\text{m}^3$), P is the pressure (in N/m^2), C is a constant that characterizes the composition and temperature of the gas (in m^2/sec^2), and Φ and Ψ are functions that characterize the hydraulic conditions for introduction of the liquid into the channel and for withdrawal of the gas from the channel.

The rate of the liquid–gas phase transition per unit volume is constant. The length of the two-phase zone is smaller than the length of the channel, and a purely gas-phase flow exists beyond this zone. Pressure and gas velocity perturbations propagate instantaneously along the length. The liquid velocity changes simultaneously throughout the length of the channel. By virtue of these assumptions, the law of conservation of momentum for the liquid and the gas reduces to (3) and (4). The liquid density and the composition and temperature of the gas are constant, and the gas is ideal, (5)–(7). The liquid arrives at the channel inlet, (8), and its flow rate depends on the pressure, (9). The gas, whose flow rate also depends on the pressure, (10), flows out at the outlet.

Steady Solution. We assume that the function $\Phi(P)$ decreases monotonically and the function $\Psi(P)$ increases monotonically in accordance with the physical meaning. This means that as the pressure grows in the channel, the inlet flow rate of the liquid decreases, while the outlet flow rate of the gas increases, and the inlet and outlet hydraulic conditions (9) and (10) uniquely determine the stationary flow rate G_s and pressure P_s:

$$\Phi(P_s) = \Psi(P_s) = G_s.$$

Here and below, the subscript s denotes the corresponding stationary quantity.

Equation (2) with boundary condition (8) reduces to the following steady equation:

$$G_s \frac{d\beta_s}{dz} = -W, \quad z \in [0, L], \quad \beta_s(0) = 1, \tag{11}$$

which determines the unique steady solution of problem (1)–(10):

$$\beta_s(z) = \begin{cases} 1 - zW/G_s, & \text{if } z \in [0, L_{\text{liquid},s}], \quad L_{\text{liquid},s} = G_s/W, \\ 0, & \text{if } z \in [L_{\text{liquid},s}, L], \quad L_{\text{gas},s} = L - L_{\text{liquid},s}, \end{cases}$$

$$u_{\text{liquid},s} = \frac{G_s}{\rho_{\text{liquid}}}, \quad \rho_{\text{gas},s} = \frac{P_s}{C}, \quad u_{\text{gas},s} = \frac{G_s}{\rho_{\text{gas},s}}, \quad \rho_s(z) = \beta_s \rho_{\text{liquid}} + (1 - \beta_s)\rho_{\text{gas},s}, \tag{12}$$

where $L_{\text{liquid},s}$ and $L_{\text{gas},s}$ are the stationary lengths of the two-phase and gas-phase zones of the channel.

Analysis of the Stability of a Steady Solution in a Linear Approximation. By virtue of conditions (5) and (6), the equality

$$\frac{\partial \rho}{\partial t} = \rho_{\text{liquid}} \frac{\partial \beta}{\partial t} + \frac{1-\beta}{C}\frac{dP}{dt} - \frac{P}{C}\frac{\partial \beta}{\partial t} = \left(\rho_{\text{liquid}} - \frac{P}{C}\right)\frac{\partial \beta}{\partial t} + \frac{1-\beta}{C}\frac{dP}{dt}$$

is fulfilled, and we can eliminate the variable ρ from Eq. (1):

$$\left(\rho_{\text{liquid}} - \frac{P}{C}\right)\frac{\partial \beta}{\partial t} + \frac{1-\beta}{C}\frac{dP}{dt} + \frac{\partial G}{\partial z} = 0. \tag{13}$$

Using conditions (3) and (5), we reduce Eq. (2) to the form

$$\rho_{\text{liquid}} \frac{\partial \beta}{\partial t} + G(0,t)\frac{\partial \beta}{\partial z} = -W. \tag{14}$$

For the gas-phase zone, which is formed after the liquid disappears, instead of (13) and (14) it is sufficient to use one equation derived from (1) with allowance for (5) and (6) for $\beta = 0$:

$$\frac{1}{C}\frac{dP}{dt} + \frac{\partial G}{\partial z} = 0. \tag{15}$$

We linearize Eqs. (13)–(15) with respect to small perturbations of the steady solution by making the substitution

$$\beta(z,t) = \beta_s(z) + \beta''(z,t), \quad G(z,t) = G_s + G''(z,t), \quad P(t) = P_s + P''(t)$$

and leaving, in the derived equations, terms that are linear with respect to the perturbations β'', G'', and P''. As a result, we obtain a system of two equations for the two-phase zone of the channel ($0 < z < L_{\text{liquid},s}$ and

$t > 0$):

$$(\rho_{\text{liquid}} - \rho_{\text{gas},s}) \frac{\partial \beta''}{\partial t} + \frac{(1-\beta_s)}{C} \frac{dP''}{dt} + \frac{\partial G''}{\partial z} = 0; \tag{16}$$

$$\rho_{\text{liquid}} \frac{\partial \beta''}{\partial t} + G_s \frac{\partial \beta''}{\partial z} + G''(0,t) \frac{d\beta_s}{dz} = 0 \tag{17}$$

and one equation for the gas-phase zone of the channel ($L_{\text{liquid},s} < z < L$ and $t > 0$):

$$\frac{1}{C} \frac{dP''}{dt} + \frac{\partial G''}{\partial z} = 0. \tag{18}$$

We add the linearized boundary conditions (8)–(10) to this system:

$$z = 0: \qquad \beta'' = 0; \tag{19}$$

$$z = 0: \qquad G'' = \frac{d\Phi}{dP}(P_s)P''; \tag{20}$$

$$z = L: \qquad G'' = \frac{d\Psi}{dP}(P_s)P''. \tag{21}$$

We transform the term $G''(0,t)$ in Eq. (17) using condition (20) and substitute the relation for the stationary profile of the volume fraction of the liquid (12) into Eq. (16). After this, we reduce system (16)–(21) to dimensionless form by using the stationary quantities $G' = G''/G_s$, $P' = P''/P_s$, and $\beta' = \beta''$ as the units of the scale of independent variables and using the length of the two-phase zone and the time of the presence of the liquid in this zone (for a steady solution) $z' = z/L_{\text{liquid},s}$ and $t' = t/T_{\text{liquid},s}$, where $L_{\text{liquid},s} = G_s/W_s$ and $T_{\text{liquid},s} = L_{\text{liquid},s}/u_{\text{liquid},s} = \rho_{\text{liquid}}/W$, as the units of the scale of length and time, respectively. As a result, we write the linear homogeneous system of partial differential equations for dimensionless perturbations of the steady solution for $t' > 0$:

$$(1-\varepsilon) \frac{\partial \beta'}{\partial t'} + \varepsilon z' \frac{dP'}{dt'} + \frac{\partial G'}{\partial z'} = 0, \qquad z' \in [0,1]; \tag{22}$$

$$\frac{\partial \beta'}{\partial t'} + \frac{\partial \beta'}{\partial z'} + h_0 P' = 0, \qquad z' \in [0,1]; \tag{23}$$

$$\varepsilon \frac{dP'}{dt'} + \frac{\partial G'}{\partial z'} = 0, \qquad z' \in [1, 1+d]. \tag{24}$$

The boundary conditions are as follows:

$$z' = 0: \qquad \beta' = 0; \tag{25}$$

$$z' = 0: \qquad G' = -h_0 P'; \tag{26}$$

$$z' = 1 + d: \qquad G' = h_1 P'. \tag{27}$$

System (22)–(27) includes four nonnegative constant dimensionless variables:

$$\varepsilon = \frac{\rho_{\text{gas},s}}{\rho_{\text{liquid}}}, \quad d = \frac{L_{\text{gas},s}}{L_{\text{liquid},s}}, \quad h_0 = \frac{-d\Phi}{dP}(P_s)\frac{P_s}{G_s}, \quad h_1 = \frac{d\Psi}{dP}(P_s)\frac{P_s}{G_s}. \tag{28}$$

We shall perform a stability analysis of the zero steady solution of problem (22)–(27) by making the following substitution: $\beta'(z',t') = \beta^*(z')\exp(\lambda t')$, $G'(z',t') = G^*(z')\exp(\lambda t')$, and $P'(t') = P^*\exp(\lambda t')$, where all the quantities, except for z' and t', are considered in the complex plane, the vector function (β^*, G^*, P^*) of the variable z is the eigenfunction for problem (22)–(27), and the parameter λ is the eigenvalue (the spectral point). Finally, we obtain the spectral problem that consists in finding λ for which there are nonzero solutions $\beta^*(z)$, $G^*(z)$, P^* of the system

$$\frac{dG^*}{dz'} + (1-\varepsilon)\lambda\beta^* + \varepsilon\lambda z' P^* = 0, \qquad z' \in [0,1]; \tag{29}$$

$$\frac{d\beta^*}{dz'} + \lambda\beta^* + h_0 P^* = 0, \qquad z' \in [0,1]; \tag{30}$$

$$\frac{dG^*}{dz'} + \varepsilon\lambda P^* = 0, \qquad z' \in [1, 1+d]. \tag{31}$$

The boundary conditions are of the form

$$z' = 0: \quad \beta^* = 0; \tag{32}$$

$$z' = 0: \quad G^* = -h_0 P^*; \tag{33}$$

$$z' = 1 + d: \quad G^* = h_1 P^*. \tag{34}$$

We integrate Eq. (30) for the two-phase zone over the length of this zone with the initial condition (32), regarding λ and P^* as the desired parameters:

$$\beta^*(z') = -\int_0^{z'} h_0 P^* \exp(\lambda(z - z'))\, dz = h_0 P^* \frac{\exp(-\lambda z') - 1}{\lambda}. \tag{35}$$

With allowance for the dependence (35), Eq. (29) reduces to the form

$$\frac{dG^*}{dz'} = -[(1-\varepsilon)h_0(\exp(-\lambda z') - 1) + \varepsilon\lambda z']P^* = 0. \tag{36}$$

Integrating Eq. (36) with the initial condition (33) over the length of the two-phase zone, we obtain

$$G^*(1) = -h_0 P^* + (1-\varepsilon)h_0\left(\frac{\exp(-\lambda) - 1}{\lambda} + 1\right)P^* - \frac{\varepsilon\lambda P^*}{2}. \tag{37}$$

Integrating Eq. (31) over the length of the gas-phase zone, we write

$$G^*(1+d) = C^*(1) - \varepsilon d\lambda P^*. \tag{38}$$

Having substituted (37) into (38), we have an equation that relates the variables G^* and P^* at the channel outlet:

$$G^*(1+d) = -h_0 P^* + (1-\varepsilon)h_0\left(\frac{\exp(-\lambda) - 1}{\lambda} + 1\right)P^* - \varepsilon\lambda P^*(1/2 + d). \tag{39}$$

Since condition (34) must be satisfied at the channel's outlet, the problem of the existence of nonzero solutions of problem (29)–(34) reduces to the condition of vanishing of the determinant of the system of two linear homogeneous equations (34) and (39) in the unknowns P^* and $G^*(1+d)$:

$$F = h_1 + h_0 - (1-\varepsilon)h_0\left(\frac{\exp(-\lambda) - 1}{\lambda} + 1\right) + \varepsilon\lambda(1/2 + d) = 0. \tag{40}$$

Thus, we have derived the (complex) spectral equation (40) of the form $F(\lambda) = 0$, the existence of at least one root λ of which in the right-hand half of the complex plane indicates instability of the steady solution of the initial system (1)–(10) in the linear approximation. Equation (40) includes four dimensionless parameters h_0, h_1, ε, and d that affect the stability in the general case.

Parabolic Analysis of Instability Regions. A numerical analysis by the method [3] of location of spectral points, i.e., the roots λ of Eq. (40) on the complex plane, shows that there are certain regions of the parameters h_0, h_1, ε, and d for which a pair of complex-conjugate numbers $\lambda = r \pm i\omega$ is positioned in the right-hand half of the complex plane ($r > 0$), which means instability of the steady solution in the linear approximation. Numerical integration of the nonlinear boundary-value problem (1)–(10) by the fitting method [4] shows that in these ranges of the parameters an arbitrary perturbation of the steady solution leads to establishment of self-oscillations of constant frequency and constant amplitude. Figure 1 shows the variation of the dimensionless liquid flow rate G' at the entrance to the channel and of the dimensionless gas flow rate at the exit from the channel (curves 1 and 2, respectively), in relation to the dimensionless time t' for $\varepsilon = 0.02$, $h_1 = 1$, $h_0 = 6$, and $d = 10$. The oscillation amplitude tends smoothly to zero in approach to the parametric stability boundary, i.e., one can speak of smooth excitation of oscillations.

We shall clarify the physical meaning of the parameters h_0 and h_1 in a number of particular cases. If the arrival of the liquid at the channel inlet is described by the law $G = \Phi(P) = ((P_0 - P)/k_0)^{1/2}$, where k_0 is the coefficient of inlet hydraulic resistance and P_0 is the constant pressure before the entrance to the

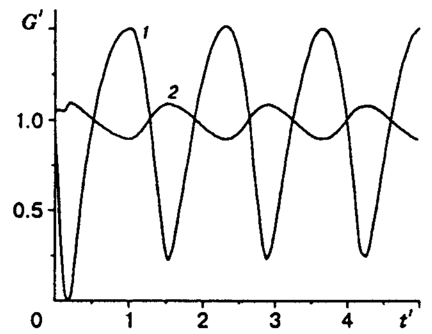

Fig. 1

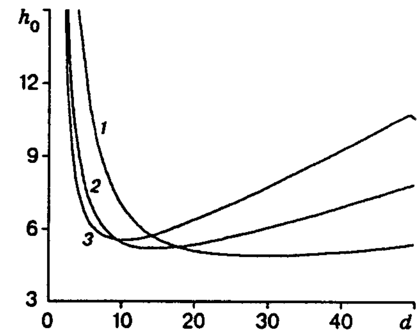

Fig. 2

channel, then, by definition (28), $h_0 = (2\Delta P_0/P_s)^{-1}$, where $\Delta P_0 = P_0 - P_s$, i.e., the parameter h_0 is inversely proportional to the doubled stationary pressure difference at the inlet to the channel ΔP_0. For subcritical gas outflow from the channel outlet, we have $G = \Psi(P) = ((P - P_1)/k_1)^{1/2}$, where k_1 is the coefficient of outlet hydraulic resistance and P_1 is the constant pressure after the gas leaves the channel, and, by definition (28), $h_1 = (2\Delta P_1/P_s)^{-1}$, where $\Delta P_1 = P_s - P_1$, i.e., the parameter h_1 is inversely proportional to the doubled stationary pressure difference at the exit from the channel ΔP_1. For critical gas outflow at the outlet, we have $G = \Psi(P) = P/k_1$, and hence $h_1 = 1$.

Figure 2 shows the boundaries of the instability regions in the plane of parameters (d, h_0) for $h_1 = 1$ and $\varepsilon = 0.01$, 0.02, and 0.03 (curves 1–3), the instability regions being located above the boundaries. The boundaries correspond to the case where the right pair of complex-conjugate roots of the spectral equation (40) is on the imaginary axis. Instability always arises at a rather large value of h_0 (a small pressure difference at the channel outlet). For a given ratio of the phase densities, for example, $\varepsilon = 0.02$, there is a critical level of the parameter $h_0 = 5$ below which oscillations do not occur for $h_0 < 5$. Slightly above this level at $h_0 > 5$ oscillations appear for the ratio $d = 15$ between the lengths of the gas–phase and two-phase zones of the channel. Upon further increase in h_0, for example, for $h_0 = 6$, the instability interval in the parameter d expands ($8 < d < 30$). In the general case the instability corresponds to a fairly large, or, conversely, to a fairly small, value of d. For example, when the length of the two-phase zone is equal to the channel length ($d = 0$), the steady solution is stable, independently of the value of h_0 and ε. We note that the extension of the stability region relative to h_0 with a large increase in the parameter d corresponds to the results of an analysis of a single-velocity (pseudohomogeneous) model of a two-phase flow [3]. At the same time, the extension of the stability region relative to h_0 with a large increase in the parameter d is unexpected at first sight. Probably, this is explained by the hypothesis of fluid motion in the form of a solid jet that was introduced into the model. In this case, in contrast to the pseudohomogeneous model, shortening of the gas-phase segment can ensure stability, because the time lag between the variation in the outlet gas flow rate and the variation in the inlet liquid flow rate, which is necessary for the occurrence of oscillations decreases. In the pseudohomogeneous model, such a lag is provided by the two-phase segment itself.

REFERENCES

1. M. S. Natanzon, *Combustion Instability* [in Russian], Mashinostroenie, Moscow (1986).
2. J. A. Boure, A. E. Bergles, and L. S. Tong, "Review of two-phase flow instability," *Nucl. Eng. Design*, **25**, 165–192 (1973).
3. A. N. Stegasov, A. B. Shigarov, and V. A. Kirillov, "Analytical method of studying unstable operation of reactors with a liquid–gas phase transition," *Teor. Osn. Khim. Technol.*, **29**, No. 5, 475–481 (1995).
4. A. B. Shigarov, A. N. Stegasov, and V. A. Kirillov, "Method of calculating transient and oscillatory regimes in two-phase flows with a phase transition, " *Teor. Osn. Khim. Tekhnol.*, **25**, No. 4, 524–532 (1991).